Decoherence ensures classicality beyond the Ehrenfest time as $\hbar \to 0$

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In closed quantum systems, wavepackets can spread exponentially in time due to chaos, forming long-range superpositions in just seconds for ordinary macroscopic systems. A weakly coupled environment is conjectured to decohere the system and restore the quantum-classical correspondence while necessarily introducing diffusive noise—but for what coupling strength, and under what conditions? For Markovian open systems with Hamiltonians of the form $\hat{H} = \hat{p}^2/2m + V(\hat{x})$ and linear Lindblad operators, we prove the quantum and classical evolutions are close whenever the strength of the environment-induced diffusion exceeds a threshold $\hbar^{4/3} A_c$, were A_c is a characteristic scale of the classical dynamics. (A companion paper treats more general Hamiltonians and Lindblad operators.) The bound applies for all observables and for times exponentially longer than the Ehrenfest timescale, which is when the correspondence can break down in closed systems. The strength of the diffusive noise can vanish in the classical limit to give the appearance of reversible dynamics. The 4/3 exponent may be optimal, as Toscano et al. have found evidence that the quantum-classical correspondence breaks down in some systems when the diffusion is any weaker.

We study the macroscopic emergence of classical mechanics from quantum mechanics in the limit when $\hbar$ is small compared to the characteristic features of the system. This is a well-studied topic in the case of *closed* quantum systems – that is to say, in Hamiltonian systems isolated from any influence from an external environment – where classical and quantum observables are known [1–6] by Egorov's theorem [1] to match closely for times up to the Ehrenfest time $\tau_{\mathrm{E}} \sim \lambda_{\mathrm{L}}^{-1} \log(S_{\mathrm{c}}/\hbar)$. This timescale is governed by the dominant Lyapunov exponent λ_{L} and the characteristic action scale S_{c} of the classical dynamics, and it quantifies the time for a minimal uncertainty wavepacket to spread significantly due to chaos. The above correspondence arises essentially because of Ehrenfest's theorem [7], which implies that a localized wavepacket will approximately follow a classical equation of motion as long as it remains well-localized.

For real macroscopic systems, the Ehrenfest time can be quite short—even seconds or minutes—because the dependence on $\hbar^{-1}$ is only logarithmic [8–10]. In closed systems beyond the Ehrenfest time, even as $\hbar \to 0$, the correspondence between the classical and quantum evolution breaks down. In particular,

(1) superpositions over macroscopic distances are generated, detectable through delicate interference experiments [5, 11–16] (cf. [17–20]); relatedly, the Wigner function develops negativity [9, 21–24] (cf. [25, 26]).

(2) large differences possibly arise between quantum expectation values and the corresponding classical predictions even for smooth observables like $\hat{x}^2$, according to some numerical studies [27, 28] (cf. [29]).

In other words, the $\hbar \to 0$ and $t \to \infty$ limits do not commute in closed systems: if one fixes a time duration $t > 0$ and takes $\hbar \to 0$, the quantum state trajectory approaches the classical state trajectory, but if one fixes arbitrarily small $\hbar > 0$ and takes $t \to \infty$, then the quantum trajectory may develop superpositions over macroscopic distances. (The limit is *singular* in the sense of Berry [16, 30–32].)

Despite this theoretical breakdown, macroscopic systems appear to obey the laws of classical mechanics for much longer times. To theoretically jus-

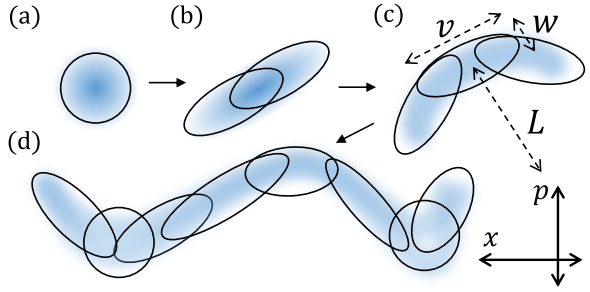

FIG. 1. (a) An initial pure quantum Gaussian state $\rho(t{=}0)$ evolves in phase space. (b) At short times the dynamics admit a local harmonic approximation, squeezing the distribution via classical flow and broadening it via diffusion D. The state, now mixed, can be approximated by a mixture $\tilde{\rho}(t)$ of pure squeezed Gaussian states (ellipses). (c) Over intermediate times, the true distribution $\rho(t)$ bends with radius of curvature L due to anharmonicity, while maintaining minimum thickness $\sqrt{D/\lambda_{\mathrm{L}}}$, for classical local Lyapunov exponent λ_{L}. Then $\rho(t)$ can still be approximated by $\tilde{\rho}(t)$ if the distribution $\rho(t)$ can be snugly covered by ellipses of width w and length $v = \hbar/w$, which geometrically requires $v \lesssim \sqrt{wL}$ and $w \lesssim \sqrt{D/\lambda_{\mathrm{L}}}$, or $D \gtrsim \hbar^{4/3} \lambda_{\mathrm{L}} L^{-2/3}$. (d) Over long times the squeezing of the Gaussian decomposition varies locally but remains bounded as $\rho(t)$ spreads over the accessible phase space.

tify a quantum-classical correspondence beyond the Ehrenfest time, one longstanding suggestion is to consider decoherence effects from the environment, acknowledging that macroscopic systems are rarely well-isolated [12, 13, 15, 33, 34] (cf. [17–19, 29, 35–41]). Numerical simulations and analytical arguments suggest that decoherence successfully restores the quantitative agreement between the quantum and classical evolution of many specific systems [14, 42–45]. In particular, it was conjectured that there is a regime where the system-bath coupling is (i) large enough for decoherence to ensure classicality by inhibiting long-range coherence in phase space, but still (ii) small enough that the noise introduced by the bath does not significantly alter the classical dynamics on fixed time scales [46], so the system *appears* well-isolated [43–45].

In this work we prove several aspects of the quantum-classical correspondence suggested above using decoherence modeled by the Lindblad equation. The results presented here apply to Hamiltonians of the common form $\hat{H} = \hat{p}^2/2m + V(\hat{x})$ with linear Lindblad operators and no friction. Such a system may represent multiple particles in multiple spatial dimensions, with general position-dependent interactions [47]. (With N particles in n spatial dimensions, we have $d = Nn$ degrees of freedom.)

The case of general Hamiltonians $\hat{H}(\hat{x}, \hat{p})$ and Lindblad operators $\hat{L}_k(\hat{x}, \hat{p})$ will be presented in a companion paper. The linear Lindblad operators used here may be seen as the first-order approximation of a general environmental interaction, and their effect is to introduce both decoherence and noise. The noise manifests as diffusion in phase space, or "environment-induced diffusion."

We prove that given sufficient diffusion, for times $t \ll \hbar^{-1/2}$, the quantum evolution under the Lindblad equation is well-approximated by the above classical evolution for all possible observables, in a sense we will make precise. This timescale is exponentially longer than the Ehrenfest time. Indeed, for a macroscopic system with characteristic action scale $S_c \sim 1\,\mathrm{kg}\cdot\mathrm{m}^2/\mathrm{s}$ and Lyapunov exponent $\lambda_L \sim 1\,\mathrm{s}^{-1}$, the Ehrenfest timescale is $\tau_E \sim \lambda_L^{-1}\log(S_c/\hbar) \sim 1$ minute, while the bound presented below extends to the timescale $\tau \sim \lambda_L^{-1}\sqrt{S_c/\hbar} \sim 10^{17}\,\mathrm{s} \sim 1$ billion years.

For a quantum system of finite d degrees of freedom, assuming initial state $\rho(t{=}0)$ that can be well-approximated by a mixture of coherent Gaussian states, we consider the quantum evolution $\rho(t)$ under the Lindblad equation $\partial_t\rho(t) = \hat{\mathcal{L}}[\rho(t)]$ with Lindbladian

$$\hat{\mathcal{L}}[\rho] = \frac{-i}{\hbar}[\hat{H}, \rho] + \frac{1}{\hbar}\sum_k (\hat{L}_k \rho \hat{L}_k^\dagger - \frac{1}{2}\{\hat{L}_k^\dagger \hat{L}_k, \rho\}). \tag{1}$$

We treat the case of linear Lindblad operators $\hat{L}_k = \ell_x \hat{x}^k$ and $\hat{L}_{k+d} = \ell_p \hat{p}^k$ for $k = 1, \ldots, d$. We compare the quantum evolution $\rho(t)$ to the evolution of a probability distribution $f(t)$ that obeys a corresponding classical dynamics, making use of the Wigner transform $\mathcal{W}[\cdot]$ (whose needed properties are recalled in Appendix B). The initial classical distribution is taken to be the one naturally associated to the initial quantum state: $f(t{=}0) = \mathcal{W}[\rho(t{=}0)]$. The corresponding classical dynamics are given by a frictionless Fokker-Planck equation $\partial_t f = \mathcal{L}[f]$ using the Liouvillian [48, 49]

$$\mathcal{L}[f] = -\omega^{ab}(\partial_a f)(\partial_b H) + \frac{1}{2}D^{ab}\partial_a\partial_b f, \tag{2}$$

where the first term above produces classical Hamiltonian flow using $H = p^2/2m + V(x)$, while the second term above produces diffusion with diffusion matrix

$$D = \begin{pmatrix} |\ell_p|^2 \mathbb{1}_d & 0 \\ 0 & |\ell_x|^2 \mathbb{1}_d \end{pmatrix}. \tag{3}$$

Above $\omega = \left(\begin{smallmatrix} 0 & \mathbb{1}_d \\ -\mathbb{1}_d & 0 \end{smallmatrix}\right)$ is the (antisymmetric) symplectic form and ∂_a denotes the partial derivative in phase space, where the indices $a, b \in \{\mathrm{x}_1, \ldots, \mathrm{x}_d, \mathrm{p}_1, \ldots, \mathrm{p}_d\}$ range over the $2d$ directions in phase space and repeated indices are summed.

Just as the diffusion equation arises from a random walk, the Fokker-Planck equation in (2) could also be interpreted as arising from a Langevin equation [50], describing noisy trajectories on phase space. Thus (2) describes an ensemble of noisy classical trajectories, each roughly following the classical equation of motion when D is small.

In order to state the main result, we now identify some key scales in any Hamiltonian $\hat{H} = \hat{p}^2/2m + V(\hat{x})$. At each position x there is a local harmonic approximation to the dynamics depending on the Hessian $\nabla^2 V(x)$ of the potential. The **harmonic time** $\tau_H := \sqrt{m/\|\nabla^2 V\|}$ is the shortest timescale associated with such local dynamics, where

$$\|\nabla^j V\| := \sup_x \sup_{\|w_i\|=1} \left| \left[\prod_{i=1}^{j} (w_i \cdot \nabla)\right] V(x) \right| \tag{4}$$

is the maximum over phase space of j directional derivatives of the potential. Similarly, the **aspect parameter** $\eta_H := \sqrt{m\|\nabla^2 V\|} = m/\tau_H = \|\nabla^2 V\|\tau_H$ (with units of [momentum/length] $\sim$

kg/s) picks out a preferred "aspect ratio" in phase space with respect to which a Gaussian distribution can be considered circular or unsqueezed. The **anharmonic action** $s_H := \eta_H^3/(\tau_H\|\nabla^3 V\|)^2 = m^{1/2}\|\nabla^2 V\|^{5/2}\|\nabla^3 V\|^{-2}$ is the action scale above which anharmonicities of the potential (quantified by the max *third* derivative $\|\nabla^3 V\|$) are important. (It is the unique action scale constructible from m, $\|\nabla^2 V\|$, and $\|\nabla^3 V\|$.) The **anharmonic length** $x_H := \sqrt{s_H/\eta_H} = \|\nabla^2 V\|/\|\nabla^3 V\|$ and **anharmonic momentum** $p_H := \sqrt{s_H \eta_H} = \sqrt{m\|\nabla^2 V\|}/\|\nabla^3 V\|$ are then preferred scales satisfying $x_H p_H = s_H$. Finally, the pure Gaussian quantum states with the preferred covariance matrix

$$\sigma_* := \frac{\hbar}{2}\begin{pmatrix} \eta_H^{-1}\mathbb{1}_d & 0 \\ 0 & \eta_H \mathbb{1}_d \end{pmatrix} \propto \begin{pmatrix} x_H \mathbb{1}_d & 0 \\ 0 & p_H \mathbb{1}_d \end{pmatrix} \tag{5}$$

are the (unsqueezed) **coherent states**.

We can now state our main result.

Theorem 1. *Let $\rho(t)$ solve the Lindblad equation (1) in d variables with Hamiltonian $\hat{p}^2/2m + V(\hat{x})$ and $2d$ Lindblad operators $\hat{L}_k = \ell_{\mathrm{x}}\hat{x}^k$ and $\hat{L}_{k+d} = \ell_{\mathrm{p}}\hat{p}^k$ for $k = 1, \ldots, d$, corresponding to isotropic diffusion in position and momentum with respective diffusion constants $D_{(\mathrm{x})} = \hbar|\ell_{\mathrm{p}}|^2$ and $D_{(\mathrm{p})} = \hbar|\ell_{\mathrm{x}}|^2$. Assume the initial state $\rho(t{=}0)$ is given by a mixture of Gaussian states that are squeezed relative to the coherent states by no more than a factor $z := \max\{\frac{\hbar/s_H}{D_0}, 1\} \geq 1$ (i.e., with covariance matrix σ s.t. $z^{-1}\sigma_* \leq \sigma \leq z\sigma_*$) where*

$$D_0 := \min\left\{\frac{D_{(\mathrm{x})}}{x_H^2/\tau_H}, \frac{D_{(\mathrm{p})}}{p_H^2/\tau_H}\right\} \tag{6}$$

is a dimensionless measure of the diffusion strength. Finally, let $f(t)$ solve the Fokker-Planck equation (2) with initial condition $f(0) = \mathcal{W}[\rho(0)]$, and assume the potential $V(x)$ has bounded second and third derivatives. Then there exists a quantum trajectory $\tilde{\rho}(t)$ which is a mixture of Gaussians satisfying

$$\begin{aligned} \|\widetilde{\rho}(t) - \rho(t)\|_{\mathrm{Tr}} &\leq \epsilon \\ \|\mathcal{W}[\widetilde{\rho}(t)] - f(t)\|_{L^1} &\leq \epsilon \end{aligned} \tag{7}$$

with error

$$\epsilon = d^{\frac{3}{2}}\frac{t}{\tau_H}\sqrt{\frac{\hbar}{s_H}}\max\left\{\frac{\hbar/s_H}{D_0}, 1\right\}^{\frac{3}{2}}. \tag{8}$$

Recall τ_H, s_H, x_H, p_H above are just characteristic scales set by the classical Hamiltonian. In (7), $\|\hat{A}\|_{\mathrm{Tr}} := \mathrm{Tr}[(\hat{A}^\dagger\hat{A})^{1/2}]$ is the trace norm on quantum operators and $\|f\|_{L^1} := \int \mathrm{d}\alpha |f(\alpha)|$ is the analogous L^1 norm on classical phase-space functions; as recalled in Appendix A, they constrain the probability of a discrepancy being revealed by any observation or measurement.

We turn to the proof after some interpretation. How much environment-induced diffusion is necessary for Theorem 1 to ensure a close quantum-classical correspondence? We find Eq. (7) holds with error ϵ for any

$$\epsilon \geq d^{\frac{3}{2}}\frac{t}{\tau_H}\sqrt{\frac{\hbar}{s_H}}, \tag{9}$$

if the diffusion satisfies

$$D_0 \geq \left(\frac{d^{\frac{3}{2}}t}{\epsilon\, t_H}\right)^{\frac{2}{3}}\left(\frac{\hbar}{s_H}\right)^{\frac{4}{3}}. \tag{10}$$

Alternatively, if $D_0 \gtrsim (\hbar/S_H)^{\frac{4}{3}-p}$ for some power $p > 0$, we find that the error ϵ in the correspondence (7) is small for times $t \lesssim t_H(s_H/\hbar)^q$, for power $q = \min\{\frac{1}{2}, \frac{3p}{2}\}$. Note this time range is exponentially longer than the Ehrenfest time.

See Fig. 1 for a heuristic argument suggesting the adequacy of condition (10). We illustrate how the bound might be applied to physical examples in Appendix F.

Proof of Theorem 1. We will build a trajectory $\tilde{\rho}(t)$ that approximates the true evolution $\rho(t)$ by using a mixture of Gaussian states $\hat{\tau}_{\alpha,\sigma}$, each centered at a point $\alpha = (\alpha^{\mathrm{x}}, \alpha^{\mathrm{p}}) \in \mathbb{R}^{2d}$ in phase space with covariance matrix $\sigma \in \mathbb{R}^{2d}\times\mathbb{R}^{2d}$, that evolve under a local harmonic approximation to $\hat{\mathcal{L}}$. Thus our trajectory will be defined through a time-dependent distribution $p_{\alpha,\sigma}(t) \geq 0$,

$$\tilde{\rho}(t) = \iint \mathrm{d}\alpha\mathrm{d}\sigma\, p_{\alpha,\sigma}(t)\hat{\tau}_{\alpha,\sigma}, \tag{11}$$

but with the crucial proviso that $p_{\alpha,\sigma}(t)$ only supports covariance matrices σ that are **not too squeezed** (NTS) in the sense $\sigma \leq z\sigma_*$. (The distribution $p_{\alpha,\sigma}$ is a generalization of the Glauber-Sudarshan P function [51–54], which uses a single fixed σ.) Importantly, we have assumed the true initial state $\rho(t{=}0) = \iint \mathrm{d}\alpha\mathrm{d}\sigma\, p_{\alpha,\sigma}(t{=}0)\hat{\tau}_{\alpha,\sigma}$, is a (possibly trivial) mixture of such states, so that the true trajectory and our approximation initially coincide: $\tilde{\rho}(t{=}0) = \rho(t{=}0)$.

Completing the proof requires three steps.

Step 1. Constructing the Gaussian mixture $\tilde{\rho}$: If $\tilde{\rho}(t)$ is to approximate $\rho(t)$, which satisfies the Lindblad equation (1), then we would like to find $p_{\alpha,\sigma}(t)$ such that $\partial_t\tilde{\rho}(t) \approx \hat{\mathcal{L}}[\tilde{\rho}(t)] = \iint \mathrm{d}\alpha\mathrm{d}\sigma\, p_{\alpha,\sigma}(t)\hat{\mathcal{L}}[\hat{\tau}_{\alpha,\sigma}]$. To accomplish this, we will use a *harmonic* approximation $\hat{\mathcal{L}}^{(\alpha)}$ to $\hat{\mathcal{L}}$ near the point α in phase space, where harmonic dynamics are characterized by a quadratic

Hamiltonians and linear Lindblad operators. The benefit of using this approximation is that harmonic dynamics exactly preserve the set of Gaussian states [55–57] (see also [58, 59]). We will find a function $p_{\alpha,\sigma}(t)$ that exactly solves

$$\partial_t \tilde{\rho}(t) = \iint \mathrm{d}\alpha \mathrm{d}\sigma \, p_{\alpha,\sigma}(t) \hat{\mathcal{L}}^{(\alpha)}[\hat{\tau}_{\alpha,\sigma}]. \tag{12}$$

In the present case of linear Lindblad operators and Hamiltonian $\hat{H} = \hat{p}^2/2m + V(\hat{x})$, the harmonic approximation $\hat{\mathcal{L}}^{(\alpha)}$ to $\hat{\mathcal{L}}$ at α is obtained simply by replacing the potential with its local quadratic approximation about $x = \alpha^{\mathrm{x}}$, i.e., by replacing $V(\hat{x})$ with $V^{[\alpha^{\mathrm{x}},2]}(\hat{x})$ where $V^{[\alpha^{\mathrm{x}},2]}(x) = V(\alpha^{\mathrm{x}}) + (x-\alpha^{\mathrm{x}}) \cdot \nabla V(\alpha^{\mathrm{x}}) + [(x-\alpha^{\mathrm{x}}) \cdot \nabla]^2 V(\alpha^{\mathrm{x}})$. Before we proceed to analyze the error between $\tilde{\rho}(t)$ and $\rho(t)$, we first show that a function $p_{\alpha,\sigma}(t)$ solving (12) can be found.

Under the harmonic approximation about α, a Gaussian evolves such that [55–57] its centroid α follows the classical (diffusionless) flow on phase space with flow vector

$$U(\alpha) := \begin{pmatrix} \alpha^{\mathrm{p}}/m \\ -\nabla V(\alpha^{\mathrm{x}}) \end{pmatrix} \tag{13}$$

in the sense that $\mathrm{d}\alpha(t)/\mathrm{d}t = U(\alpha)$, while the covariance matrix σ evolves by $\mathrm{d}\sigma(t)/\mathrm{d}t = S(\alpha,\sigma)$ where

$$S(\alpha,\sigma) := F(\alpha)\sigma + \sigma F^{\top}(\alpha) + D, \tag{14}$$

$$F(\alpha) := \begin{pmatrix} 0 & -\mathbb{1}_d/m \\ \nabla^2 V(\alpha^{\mathrm{x}}) & 0 \end{pmatrix}, \tag{15}$$

and where $D = \mathrm{diag}(D_{(\mathrm{x})}\mathbb{1}_d, D_{(\mathrm{p})}\mathbb{1}_d)$ is the diffusion matrix. This describes the skewing (by F) and broadening (by D) of the Gaussian; only the latter increases mixedness. Equivalently, the Gaussian obeys

$$\hat{\mathcal{L}}^{(\alpha)}[\hat{\tau}_{\alpha,\sigma}] = [U(\alpha) \cdot \partial_\alpha + S(\alpha,\sigma) \cdot \partial_\sigma] \, \hat{\tau}_{\alpha,\sigma}. \tag{16}$$

With (11) and (16), our desired condition (12) becomes

$$\begin{aligned} &\iint \mathrm{d}\alpha \mathrm{d}\sigma \, \hat{\tau}_{\alpha,\sigma} \frac{\mathrm{d}}{\mathrm{d}t} p_{\alpha,\sigma}(t) \\ &= \iint \mathrm{d}\alpha \mathrm{d}\sigma \, p_{\alpha,\sigma}(t) \, [U(\alpha) \cdot \partial_\alpha + S(\alpha,\sigma) \cdot \partial_\sigma] \, \hat{\tau}_{\alpha,\sigma}. \end{aligned} \tag{17}$$

We could integrate the right-hand side of (17) by parts in σ and α to obtain a transport equation for $p_{\alpha,\sigma}$, but we would quickly lose control of the covariance matrix σ, which could be stretched arbitrarily long by the evolution. Instead, we observe that any component of the flow in the "positive" σ direction (which increases mixedness of the state) can also be re-interpreted as diffusion in the α direction [60–65].

For any choice of decomposition $S = S_{\mathrm{D}} + S_0$ we have $S \cdot \partial_\sigma \hat{\tau}_{\alpha,\sigma} = [S_0 \cdot \partial_\sigma + \frac{1}{2} S_{\mathrm{D}} \cdot \partial_\alpha \partial_\alpha] \hat{\tau}_{\alpha,\sigma}$ by the Gaussian derivative identity $\partial_\sigma \hat{\tau}_{\alpha,\sigma} = \frac{1}{2} \partial_\alpha \partial_\alpha \hat{\tau}_{\alpha,\sigma}$ (reviewed in Appendix E 1). Plugging this into (17) and integrating by parts [66], we see that (12) is satisfied (with $p_{\alpha,\sigma}$ guaranteed to be non-negative when $S_{\mathrm{D}} \geq 0$) so long as $p_{\alpha,\sigma}(t)$ solves $\partial_t p_{\alpha,\sigma}(t) = \breve{\mathcal{L}}^{(\alpha)}[p_{\alpha,\sigma}(t)]$ with

$$\begin{aligned} \breve{\mathcal{L}}^{(\alpha)}[p_{\alpha,\sigma}] := \Big[&- \partial_\alpha \cdot U(\alpha) - \partial_\sigma \cdot S_0(\alpha,\sigma) \\ &+ \frac{1}{2} \partial_\alpha \partial_\alpha \cdot S_{\mathrm{D}}(\alpha,\sigma) \Big] p_{\alpha,\sigma} \end{aligned} \tag{18}$$

where the partial derivatives in (18) are understood to act also on $p_{\alpha,\sigma}$.

We have some limited freedom in choosing S_0 and S_{D}; that is, the decomposition of S into skewing and broadening parts is not unique. We require (i) $S_{\mathrm{D}} \geq 0$ so that $p_{\alpha,\sigma}$ undergoes non-negative diffusion, and we must choose S_0 so that both (ii) $p_{\alpha,\sigma}$ remains supported on covariance matrices of pure Gaussian states and (iii) $p_{\alpha,\sigma}$ remains supported on NTS covariance matrices. Below, we choose a decomposition fulfilling these three requirements.

It is useful to work with the rescaled matrices $\bar{X} := \sigma_*^{-1/2} X \sigma_*^{-1/2}$ for $X = \sigma, S, S_0, S_{\mathrm{D}}, D$. We also use the asymmetric $\bar{F} := \sigma_*^{-1/2} F \sigma_*^{1/2}$ (note $\pm 1/2$ exponents) so that $\bar{S} = \bar{F}\bar{\sigma} + \bar{\sigma}\bar{F}^{\top} + \bar{D}$. Then we make the choice

$$\bar{S}_0(\alpha,\sigma) := [\bar{F}(\alpha) - g(\bar{\sigma})]\bar{\sigma} + \bar{\sigma}[\bar{F}^{\top}(\alpha) - g(\bar{\sigma})], \tag{19}$$

$$\bar{S}_{\mathrm{D}}(\alpha,\sigma) := \bar{D} + [g(\bar{\sigma})\bar{\sigma} + \bar{\sigma} g(\bar{\sigma})], \tag{20}$$

satisfying $\bar{S}_0 + \bar{S}_{\mathrm{D}} = \bar{S}$, where

$$g(\bar{\sigma}) = \left(\frac{D_0 s_H}{\hbar \tau_H} \right) \frac{\bar{\sigma} - \bar{\sigma}^{-1}}{1 - z^{-2}}. \tag{21}$$

First, a bit of algebra shows that $\bar{S}_0 = \bar{S}_0{}^{\top}$ and $(\bar{\sigma}^{-1}\bar{S}_0)^{\top} = -\omega^{\top}\bar{\sigma}^{-1}\bar{S}_0\omega$ (because $\omega^{\top}\bar{\sigma}\omega = \bar{\sigma}^{-1}$). As recalled in Appendix C, this ensures that a covariance matrix evolving by S_0 remains a legal covariance matrix for a pure Gaussian state under the dynamics (18).

Next, we need to show that with these choices the distribution $p_{\alpha,\sigma}(t)$ never develops support on covariance matrices violating the NTS condition $\sigma \leq z\sigma_*$, i.e., that $\bar{\sigma} = \sigma_*^{-1/2}\sigma\sigma_*^{-1/2} \leq z\mathbb{1}_d$ for any σ such $p_{\alpha,\sigma} \neq 0$. Note that an equivalent NTS condition is $\sigma \geq z^{-1}\sigma_*$ because $\sigma/(\hbar/2)$ and $\sigma_*/(\hbar/2)$ are symplectic matrices. (See Appendix C, Eq. (C15) for an elementary demonstration.) This equivalent lower-bound condition will be preserved if $\langle v|\bar{S}_0(\alpha,\sigma)|v\rangle \geq 0$ whenever v is an eigenvector of

$\bar{\sigma}$ with eigenvalue $\lambda \le z^{-1}$ ($\bar{\sigma} v = \lambda v$). From (20) we compute

$$\langle v|\bar{S}_0(\alpha,\sigma)|v\rangle \ge -2\|\bar{F}(\alpha)\|_{\rm op} z^{-1} + 2D_0 s_H \tau_H^{-1} \ge 0 \tag{22}$$

because $z := \max\{\hbar/s_H D_0, 1\} \ge \hbar/s_H D_0$ and $\|\bar{F}(\alpha)\|_{\rm op} \le \sqrt{\|\nabla^2 V\|/m} = \tau_H^{-1}$ by (15). (Here, $\|\cdot\|_{\rm op}$ denotes the operator norm, i.e., the largest singular value of a matrix.)

This ensures that $p_{\alpha,\sigma}(t)$ is only supported on NTS covariance matrices ($z^{-1}\sigma_* \le \sigma \le z\sigma_*$) for all time. Then using $\|\bar{D}\|_{\rm op} = \|\sigma_*^{-1/2} D \sigma_*^{-1/2}\|_{\rm op} = \min\{D_{(\rm x)}\eta_H, D_{(\rm p)}/\eta_H\}/(\hbar/2) = 2D_0 s_H/\hbar\tau_H$ we have by (20) that $S_{\rm D}(\alpha,\sigma) = \sigma_*^{1/2}\bar{S}_{\rm D}(\alpha,\sigma)\sigma_*^{1/2} \ge 0$ for all allowed σ. Therefore the diffusion term in (18) ensures that $p_{\alpha,\sigma}(t) \ge 0$ for all $t \ge 0$, i.e., $\tilde{\rho}(t)$ is always a true mixture of squeezed Gaussians.

To summarize, we have constructed the trajectory $\tilde{\rho}(t)$ defined through (12) with the probability distribution $p_{\alpha,\sigma}(t)$ over covariance matrices satisfying the NTS condition $\sigma \le z\sigma_*$ defined through the dynamics $\partial_t p_{\alpha,\sigma}(t) = \breve{\mathcal{L}}^{(\alpha)}[p_{\alpha,\sigma}(t)]$ of (18) using choices (19) and (20).

Step 2. Duhamel bound: Having defined our trajectory $\tilde{\rho}$, we can compare it to the exact evolution $\rho(t)$ using the Duhamel formula:

$$\tilde{\rho}(t) - \rho(t) = \int_0^t {\rm d}s\, e^{(t-s)\hat{\mathcal{L}}} \left(\partial_s - \hat{\mathcal{L}}\right)[\tilde{\rho}(s)] \tag{23}$$

Then

$$\begin{aligned}
&\|\tilde{\rho}(t) - \rho(t)\|_{\rm Tr} && (24)\\
&= \left\| \int_0^t {\rm d}s e^{(t-s)\hat{\mathcal{L}}} \left(\partial_s - \hat{\mathcal{L}}\right)[\tilde{\rho}(s)] \right\|_{\rm Tr} && (25)\\
&\le \int_0^t {\rm d}s \left\| e^{(t-s)\hat{\mathcal{L}}} \left(\partial_s - \hat{\mathcal{L}}\right)[\tilde{\rho}(s)] \right\|_{\rm Tr} && (26)\\
&\le \int_0^t {\rm d}s \left\| \left(\partial_s - \hat{\mathcal{L}}\right)[\tilde{\rho}(s)] \right\|_{\rm Tr} && (27)\\
&= \int_0^t {\rm d}s \left\| \iint {\rm d}\alpha {\rm d}\sigma\, p_{\alpha,\sigma}(s) \left(\hat{\mathcal{L}}^{(\alpha)} - \hat{\mathcal{L}}\right)[\hat{\tau}_{\alpha,\sigma}] \right\|_{\rm Tr} && (28)\\
&\le \int_0^t {\rm d}s \iint {\rm d}\alpha {\rm d}\sigma\, p_{\alpha,\sigma}(s) \left\| \delta\hat{\mathcal{L}}^{(\alpha)}[\hat{\tau}_{\alpha,\sigma}] \right\|_{\rm Tr} && (29)\\
&\le \sup_{\sigma \le z\sigma_*} \sup_{\alpha} \left\| \delta\hat{\mathcal{L}}^{(\alpha)}[\hat{\tau}_{\alpha,\sigma}] \right\|_{\rm Tr} \int_0^t {\rm d}s && (30)\\
&= t \sup_{\sigma \le z\sigma_*} \sup_{\alpha} \left\| \delta\hat{\mathcal{L}}^{(\alpha)}[\hat{\tau}_{\alpha,\sigma}] \right\|_{\rm Tr}, && (31)
\end{aligned}$$

where (27) follows from the fact that $e^{(t-s)\hat{\mathcal{L}}}$ is a completely positive map and so cannot increase the trace norm, (28) follows from (12) and the normalization of $p_\alpha(t)$, and in (29) we have defined $\delta\hat{\mathcal{L}}^{(\alpha)} := \hat{\mathcal{L}} - \hat{\mathcal{L}}^{(\alpha)}$.

If we make the replacements $\tilde{\rho} \to \mathcal{W}[\tilde{\rho}]$, $\rho \to f$, $\hat{\tau}_{\alpha,\sigma} \to \tau_{\alpha,\sigma}$, $\hat{\mathcal{L}} \to \mathcal{L}$, and $\|\cdot\|_{\rm Tr} \to \|\cdot\|_{L^1}$ in (23–31), then identical manipulations give

$$\|\mathcal{W}[\tilde{\rho}(t)] - f(t)\|_{L^1} \le t \sup_{\sigma \le z\sigma_*} \sup_{\alpha} \left\| \delta\mathcal{L}^{(\alpha)}[\tau_{\alpha,\sigma}] \right\|_{L^1}. \tag{32}$$

Here $\delta\mathcal{L}^{(\alpha)} := \mathcal{L} - \mathcal{L}^{(\alpha)}$ where, analogously to $\hat{\mathcal{L}}^{(\alpha)}$, $\mathcal{L}^{(\alpha)}$ is the harmonic approximation to the classical Liovillian obtained by replacing $V \to V^{[\alpha^{\rm x},2]}$ in (2).

Step 3. Harmonic approximation error: The error from the harmonic approximation will be small when the spatial size of the Gaussian state is small compared to the anharmonicity of the potential V. Indeed, the only modification necessary to obtain the harmonic approximation to the dynamics involves replacing the potential with its second-order Taylor approximation, $V \to V^{[\alpha^{\rm x},2]}$, so the error is proportional to the max leading-order (i.e., third-order) correction $\Delta x^3 \|\nabla^3 V\|$ at the characteristic spatial width $\Delta x \sim \|\sigma^{\rm xx}\|_{\rm op}^{\frac{1}{2}}$ of the wavepacket with covariance matrix σ. (The xx superscript is used to denote the upper left block of σ.) More precisely, in Appendix D we prove the bounds

$$\left\| \delta\hat{\mathcal{L}}^{(\alpha)}[\hat{\tau}_{\alpha,\sigma}] \right\|_{\rm Tr} \le \mu, \quad \left\| \delta\mathcal{L}^{(\alpha)}[\tau_{\alpha,\sigma}] \right\|_{L^1} \le \mu \tag{33}$$

where

$$\mu = \sqrt{3}\, d^{\frac{3}{2}} \hbar^{-1} \|\sigma^{\rm xx}\|_{\rm op}^{\frac{3}{2}} \|\nabla^3 V\|. \tag{34}$$

In the quantum case, the $\hbar^{-1}$ factor above can be seen to arise from the $\hbar^{-1}$ in the Schrödinger equation; in the classical case, it is related to the fact that the coherent state has area $\hbar$ in phase space.

Because $\tilde{\rho}(t)$ is a mixture of Gaussian states that are NTS, the harmonic errors appearing in the quantum error (31) and classical error (32) are taken only over covariance matrices satisfying $\sigma \le z\sigma_*$, so $\|\sigma^{\rm xx}\| \le z\|\sigma_*^{\rm xx}\| = z\hbar/2\eta_H$. Thus applying (33) we get that quantum and classical errors are both upper bounded by

$$\epsilon = t\mu \le \sqrt{3}\, t\, d^{\frac{3}{2}} \hbar^{-1} (z\hbar/2\eta_H)^{\frac{3}{2}} \|\nabla^3 V\| \tag{35}$$

giving (8). □

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Appendix A: Norms

For a function $f(\alpha)$ over phase space variable $\alpha = (x, p) \in \mathbb{R}^{2d}$, the **(Lebesgue)** L^q **norm** is

$$\|f\|_{L^q} := \left(\int \mathrm{d}\alpha \, |f(\alpha)|^q \right)^{1/q}. \tag{A1}$$

Our classical error bound on the difference in two distributions f and g is stated with L^1 norm: $\|f - g\|_{L^1} := \int \mathrm{d}\alpha |f(\alpha) - g(\alpha)|$. The analogous norm on the quantum side is the **trace norm** $\|\hat{A}\|_{\mathrm{Tr}} = \mathrm{Tr}[(\hat{A}^\dagger \hat{A})^{1/2}] = \sum_i \sigma_i(\hat{A})$ of an operator $\hat{A}$, i.e., the sum of the singular values $\sigma_i(\hat{A})$.

The importance of these two norms follows from their standard variational characterization. For quantum states ρ and $\tilde{\rho}$, the respective probabilities $q_{\mathrm{q}} = \mathrm{Tr}[\hat{Q}\rho(t)]$ and $\tilde{q}_{\mathrm{q}} = \mathrm{Tr}[\hat{Q}\tilde{\rho}(t)]$ for the measurement outcome associated with a projector $\hat{Q} \leq \hat{I}$ are constrained by

$$|q_{\mathrm{q}} - \tilde{q}_{\mathrm{q}}| = |\,\mathrm{Tr}[Q(\rho(t) - \tilde{\rho}(t))]| \leq \|\hat{Q}\|_{\mathrm{op}} \, \|\rho(t) - \tilde{\rho}(t)\|_{\mathrm{Tr}} \leq \|\rho(t) - \tilde{\rho}(t)\|_{\mathrm{Tr}} \tag{A2}$$

Likewise, for classical states f and $\mathcal{W}[\tilde{\rho}(t)]$ the respecive probabilities $q_{\mathrm{c}} = \int \mathrm{d}\alpha \, Q(\alpha) f(t)$ and $\tilde{q}_{\mathrm{c}} = \int \mathrm{d}\alpha \, Q(\alpha) \mathcal{W}[\tilde{\rho}(t)]$ for the measurement of any classical indicator variable $Q(\alpha) \leq 1$ are constrained by

$$|q_{\mathrm{c}} - \tilde{q}_{\mathrm{c}}| = \left| \int \mathrm{d}\alpha \, Q(\alpha)(f(t) - \mathcal{W}[\tilde{\rho}(t)]) \right| \leq \left[\sup_\alpha |Q(\alpha)| \right] \|\rho(t) - \tilde{\rho}(t)\|_{L^1} \leq \|\rho(t) - \tilde{\rho}(t)\|_{L^1}. \tag{A3}$$

Thus, two classical states cannot be readily distinguished when they are close in L^1 norm, and two quantum states cannot be readily distinguished when they are close in trace norm, no matter what measurement is performed.

Appendix B: Wigner functions

In order to construct our classical approximation to the quantum trajectory, we use the Wigner function. The Wigner function $\mathcal{W}[\rho]$ of a density matrix ρ is a real-valued function on phase space. In this paper, we will only discuss the Wigner function of mixtures of Gaussian states. Hence we are using $\mathcal{W}[\cdot]$ essentially as a compact notation for discussing Gaussian states and Gaussian distributions, without the Wigner functions playing a key role in the argument. In particular, we only need to know three facts about the Wigner function:

1. When ρ is a Gaussian state, $\mathcal{W}[\rho]$ is a Gaussian probability distribution on phase space with the same respective mean and variance: $\mathcal{W}[\hat{\tau}_{\alpha,\sigma}] = \tau_{\alpha,\sigma}$.
2. When ρ is a mixture of Gaussian states, $\mathcal{W}[\rho]$ is a mixture of the respective Gaussian distributions, because $\mathcal{W}[\cdot]$ is a linear mapping.
3. The Weyl trace formula in the special case of the expectation value of a function V of position for a Gaussian state

$$\mathrm{Tr}[\hat{\tau}_{\alpha,\sigma} V(\hat{x})] = \int \mathrm{d}\alpha \, \tau_{\alpha,\sigma}(\alpha) V(\alpha^{\mathrm{x}}). \tag{B1}$$

For completeness, we note that for a more general density matrix ρ, the Wigner function is defined as

$$\mathcal{W}[\rho](\alpha) := (2\pi\hbar)^d \int_{\mathbb{R}^{2d}} \mathrm{d}\chi \; \mathrm{Tr}\left[e^{i\chi \cdot (\hat{r} - \alpha)} \rho \right], \tag{B2}$$

$$= \int_{\mathbb{R}^d} \mathrm{d}y \, e^{iy \cdot p/\hbar} \langle x + y/2 | \rho | x - y/2 \rangle. \tag{B3}$$

where $\hat{r} = (\hat{x}, \hat{p})$ and $|y\rangle$ denotes a position eigenstate. In general it takes both positive and negative values, often called a "quasiprobability distribution." For more about the Wigner function, see e.g. the review in [67].

Appendix C: Symplectic and Hamiltonian matrices

In this section we recall basic facts about symplectic and Hamiltonian matrices. Our starting point is the symplectic form

$$\omega = \begin{pmatrix} 0 & \mathbb{1}_d \\ -\mathbb{1}_d & 0 \end{pmatrix} \tag{C1}$$

associated with the phase space $\mathbb{R}^{2d}$ for d classical degrees of freedom. A **symplectic** matrix A satisfies

$$A^\top \omega A = \omega. \tag{C2}$$

Note that the symplectic matrices $\mathrm{Sp}(2d, \mathbb{R})$ form a group which is a subgroup of the special linear group $\mathrm{SL}(2d, \mathbb{R})$.

For an one-parameter family $A(T)$ of symplectic matrices,

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}t}(A^\top \omega A) \\ &= \dot{A}^\top \omega A + A^\top \omega \dot{A}. \end{aligned} \tag{C3}$$

Taking a family starting at the origin, $A(0) = \mathbb{1}_{2d}$, we see that Lie algebra $\mathrm{sp}(2d, \mathbb{R})$ for the symplectic group $\mathrm{Sp}(2d, \mathbb{R})$ is the set of Hamiltonian matrices, which satisfy

$$F^\top \omega + \omega F = 0. \tag{C4}$$

(Hamiltonian matrices should not to be confused with the classical Hamiltonian variable [energy] or the quantum Hamiltonian operator.) Using $\omega^2 = \mathbb{1}_{2d}$ and $\omega^\top = -\omega$, we can rearrange this as

$$F^\top = \omega F \omega = -\omega^\top F \omega. \tag{C5}$$

Using the identities $\omega = A^\top \omega A = A^{-\top} \omega A^{-1}$ we can see that if F is Hamiltonian and A is symplectic, then $A^{-1} F A$ is also symplectic.

When $A(0)$ is not the identity we can rearrange (C3) by left-multiplying by $A^{-\top}$ and right-multiplying by A^{-1}:

$$A^{-\top} \dot{A}^\top \omega + \omega \dot{A} A^{-1} = 0. \tag{C6}$$

Thus $\dot{A}A^{-1}$ is Hamiltonian when $A(t)$ is symplectic. Conjugating by a A we can see that $A^{-1}\dot{A}$ is also Hamiltonian in this case.

When A is additionally symmetric, it follows that $\dot{A}$ is also symmetric so we can write

$$\dot{A} = FA + AF^\top \tag{C7}$$

for some Hamiltonian matrix F.

In order that σ is the covariance matrices of a pure Gaussian state, it most be positive definite ($\sigma > 0$), and $\sigma/(\hbar/2)$ must be symplectic ($\sigma^\top \omega \sigma = (\hbar/2)^2 \omega$) [68, 69]. As can be checked, the same is true for

$$\sigma_* := \frac{\hbar}{2} \begin{pmatrix} \eta_H^{-1} \mathbb{1}_d & 0 \\ 0 & \eta_H \mathbb{1}_d \end{pmatrix}. \tag{C8}$$

In the proof of Theorem 1 we work with such covariance matrices σ that additionally satisfy the matrix inequality

$$\sigma \leq z \sigma_*. \tag{C9}$$

for some $z \geq 1$. Inverting this (and using the positivity of σ and σ_*) we obtain the inequality

$$\sigma^{-1} \geq z^{-1} \sigma_*^{-1}. \tag{C10}$$

Now rewriting the symplectic conditions on $\sigma/(\hbar/2)$ and $\sigma_*/(\hbar/2)$ as

$$\sigma^{-1} = (\hbar/2)^{-2}\omega^\top\sigma\omega, \tag{C11}$$
$$\sigma_*^{-1} = (\hbar/2)^{-2}\omega^\top\sigma_*\omega, \tag{C12}$$

we derive from (C10) the inequality

$$\omega^\top\sigma\omega \geq z^{-1}\omega^\top\sigma_*\omega. \tag{C13}$$

De-conjugating by the symplectic form ω gives our desired equivalent NTS condition:

$$\sigma \geq z^{-1}\sigma_* \tag{C14}$$

That is, we conclude that for pure Gaussian states being "not too long" is equivalent to being "not too thin":

$$\sigma \geq z^{-1}\sigma_* \iff \sigma \leq z\sigma_*. \tag{C15}$$

More generally, for a pure Gaussian state, the eigenvalues of the covariance matrix σ come in pairs λ and $(\hbar/2)^2\lambda^{-1}$ [68, 69].

Appendix D: Harmonic approximation error bound

Lemma 1 (Error in harmonic approximation)**.** *For a Lindblad equation (1) with Hamiltonian $\hat{H} = \hat{p}^2/2m + V(\hat{x})$ and linear lindblad operators $\hat{L}_k$, the error $\delta\hat{\mathcal{L}}^{(\alpha)} := \hat{\mathcal{L}} - \hat{\mathcal{L}}^{(\alpha)}$ for the local harmonic approximation $\hat{\mathcal{L}}^{(\alpha)}$ to the quantum dynamics at α acting on the pure Gaussian quantum state $\hat{\tau}_{\alpha,\sigma}$ satisfies*

$$\left\|\delta\hat{\mathcal{L}}^{(\alpha)}[\hat{\tau}_{\alpha,\sigma}]\right\|_{\mathrm{Tr}} \leq \sqrt{\frac{5d^3}{3}}\|\nabla^3 V\|\frac{\|\sigma^{\mathrm{xx}}\|_{\mathrm{op}}^{\frac{3}{2}}}{\hbar} \tag{D1}$$

Likewise for a Fokker-Planck equation (2), the error $\delta\mathcal{L}^{(\alpha)} := \mathcal{L} - \mathcal{L}^{(\alpha)}$ for the local harmonic approximation $\mathcal{L}^{(\alpha)}$ to the classical dynamics at α acting on the Gaussian classical state $\tau_{\alpha,\sigma}$ satisfies

$$\left\|\delta\mathcal{L}^{(\alpha)}[\tau_{\alpha,\sigma}]\right\|_{L^1} \leq \sqrt{3d^3}\|\nabla^3 V\|\frac{\|\sigma^{\mathrm{xx}}\|_{\mathrm{op}}^{\frac{2}{3}}}{\hbar} \tag{D2}$$

Proof. We first bound the quantum error

$$\left\|\delta\hat{\mathcal{L}}^{(\alpha)}[\hat{\tau}_{\alpha,\sigma}]\right\|_{\mathrm{Tr}} = \left\|(\hat{\mathcal{L}} - \hat{\mathcal{L}}^{(\alpha)})[\hat{\tau}_{\alpha,\sigma}]\right\|_{\mathrm{Tr}} = \left\|-\frac{i}{\hbar}\left[\delta\hat{H}^{[\alpha,2]}, \hat{\tau}_{\alpha,\sigma}\right]\right\|_{\mathrm{Tr}} \leq \frac{2}{\hbar}\left\|\delta\hat{H}^{[\alpha,2]}\hat{\tau}_{\alpha,\sigma}\right\|_{\mathrm{Tr}} \tag{D3}$$

for the Gaussian quantum state $\hat{\tau}_{\alpha,\sigma} = |\alpha,\sigma\rangle\langle\alpha,\sigma|$ with covariance matrix σ and mean α. Here, $\hat{I}$ denotes the identity operator and $\delta\hat{H}^{[\alpha,2]} = \hat{H} - \hat{H}^{[\alpha,2]} = \delta V^{[\alpha^{\mathrm{x}},2]}(\hat{x})$ is the operator error from the harmonic approximation. By Taylor's theorem we have the classical remainder from the quadratic approximation

$$\delta V^{[\alpha^{\mathrm{x}},2]}(\alpha^{\mathrm{x}} + \Delta x) = V(\alpha^{\mathrm{x}} + \Delta x) - V^{[\alpha^{\mathrm{x}},2]}(\alpha^{\mathrm{x}} + \Delta x) = \frac{1}{3!}\left[(\Delta x \cdot \nabla)^3 V\right](\alpha^{\mathrm{x}} + \xi\Delta x) \tag{D4}$$

for some choice of $\xi \in [0,1]$ (depending on α^{x} and Δx).

Recalling that $\||\psi\rangle\langle\phi|\|_{\mathrm{Tr}}^2 = \|\psi\|^2\|\phi\|^2$, we have

$$\left\|\delta\hat{\mathcal{L}}^{(\alpha)}[\hat{\tau}_{\alpha,\sigma}]\right\|_{\mathrm{Tr}}^2 \leq \frac{4}{\hbar^2}\left\|\delta\hat{H}^{[\alpha,2]}|\alpha,\sigma\rangle\right\|^2 \tag{D5}$$
$$= \frac{4}{\hbar^2}\,\mathrm{Tr}\left[\hat{\tau}_{\alpha,\sigma}(\delta V^{[\alpha^{\mathrm{x}},2]}(\hat{x}))^2\right] \tag{D6}$$
$$= \frac{4}{\hbar^2}\int \mathrm{d}\beta\tau_{\alpha,\sigma}(\alpha+\beta)\left[\delta V^{[\alpha^{\mathrm{x}},2]}(\alpha^{\mathrm{x}} + \beta^{\mathrm{x}})\right]^2 \tag{D7}$$

where $\tau_{\alpha,\sigma}(\alpha+\beta) = \mathcal{W}[\hat{\tau}_{\alpha,\sigma}](\alpha+\beta) = \exp(-\beta^a \sigma^{-1}_{ab} \beta^b/2)/[(2\pi)^d\sqrt{\det\sigma}]$ (a positive-valued function on phase space) is the Wigner function of the pure Gaussian state $\hat{\tau}_{\alpha,\sigma} = |\alpha,\sigma\rangle\langle\alpha,\sigma|$. In Eq. D7 we have made use of the Weyl trace formula in the particularly simple case of the expectation value of an operator that is a function of position: $\mathrm{Tr}[\rho V(\hat{x})] = \int \mathrm{d}\alpha\, \mathcal{W}[\rho](\alpha) V(\alpha^{\mathrm{x}})$. Next we apply the approximation (D4) from Taylor's theorem:

$$\left\|\delta\hat{\mathcal{L}}^{(\alpha)}[\hat{\tau}_{\alpha,\sigma}]\right\|_{\mathrm{Tr}}^2 \le \frac{4}{\hbar^2}\int \mathrm{d}\beta \tau_{\alpha,\sigma}(\alpha+\beta)\left[\frac{1}{3!}[(\beta^{\mathrm{x}}\cdot\nabla)^3 V](\alpha^{\mathrm{x}}+\xi(\beta^{\mathrm{x}})\beta^{\mathrm{x}})\right]^2 \tag{D8}$$

$$\le \frac{\|\nabla^3 V\|^2}{9\hbar^2}\int \mathrm{d}\beta \tau_{\alpha,\sigma}(\alpha+\beta)|\beta^{\mathrm{x}}|^6 \tag{D9}$$

$$= \frac{\|\nabla^3 V\|^2}{9\hbar^2}\int \mathrm{d}\beta \tau_{\alpha,\sigma}(\alpha+\beta)(\beta^\top P_{\mathrm{x}}\beta)^3 \tag{D10}$$

$$= \frac{\|\nabla^3 V\|^2}{9\hbar^2}\left[(\mathrm{Tr}\,\sigma^{\mathrm{xx}})^3 + 6(\mathrm{Tr}\,\sigma^{\mathrm{xx}})\,\mathrm{Tr}((\sigma^{\mathrm{xx}})^2) + 8\,\mathrm{Tr}((\sigma^{\mathrm{xx}})^3)\right] \tag{D11}$$

$$\le \frac{\|\nabla^3 V\|^2}{9\hbar^2}\|\sigma^{\mathrm{xx}}\|_{\mathrm{op}}^3\left(d^3+6d^2+8d\right) \tag{D12}$$

$$\le \frac{5\|\nabla^3 V\|^2}{3\hbar^2}\|\sigma^{\mathrm{xx}}\|_{\mathrm{op}}^3 d^3 \tag{D13}$$

where $P_{\mathrm{x}} := \left(\begin{smallmatrix}\mathbb{1}_d & 0\\ 0 & 0\end{smallmatrix}\right)$ projects onto the x block. In (D9) we have used the Cauchy-Schwartz inequality, in (D11) we have performed the Gaussian integral (as recalled in Appendix E 2) and in (D12) we have used $\mathrm{Tr}[A^n] \le \|A\|_{\mathrm{op}}^n d^n$ for $d\times d$ positive semidefinite matrix A. Eq. (D12) implies Eq. (D1) because $d \ge 1$.

Now we turn to the classical harmonic error:

$$\left\|\delta\mathcal{L}^{(\alpha)}[\tau_{\alpha,\sigma}]\right\|_{L^1} = \left\|(\mathcal{L}-\mathcal{L}^{(\alpha)})[\tau_{\alpha,\sigma}]\right\|_{L^1} = \left\|-\omega^{ab}\partial_a(\tau_{\alpha,\sigma}\partial_b\delta H^{[\alpha,2]})\right\|_{L^1} = \left\|(\nabla_{\mathrm{p}}\tau_{\alpha,\sigma})\cdot(\nabla_{\mathrm{x}}\delta V^{[\alpha^{\mathrm{x}},2]})\right\|_{L^1} \tag{D14}$$

$$= \left\|-\tau_{\alpha,\sigma}(\sigma^{-1}\beta)^{\mathrm{p}}\cdot(\nabla_{\mathrm{x}}\delta V^{[\alpha^{\mathrm{x}},2]})\right\|_{L^1} \tag{D15}$$

$$= \|\tau_{\alpha,\sigma}^{1/2}\|_{L^2}^2\,\|\tau_{\alpha,\sigma}^{1/2}(\sigma^{-1}\beta)^{\mathrm{p}}\cdot(\nabla_{\mathrm{x}}\delta V^{[\alpha^{\mathrm{x}},2]})\|_{L^2}^2 \tag{D16}$$

where the Gaussian derivative $\partial_a\tau_{\alpha,\sigma} = -\sigma^{-1}_{ab}\beta^b\tau_{\alpha,\sigma}$ is recalled in Appendix E 1 and where in the last line we have used the Cauchy-Schwartz inequality. (Here, β in the norm is understood to represent the function $f(\alpha+\beta)=\beta$.) Then because $\|\tau_{\alpha,\sigma}^{1/2}\|_{L^2}^2 = \int \mathrm{d}\beta\,|\tau_{\alpha,\sigma}(\alpha+\beta)| = 1$ we have

$$\left\|\delta\mathcal{L}^{(\alpha)}[\tau_{\alpha,\sigma}]\right\|_{L^1}^2 = \int \mathrm{d}\beta\,\tau_{\alpha,\sigma}(\alpha+\beta)\left|(\sigma^{-1}\beta)^{\mathrm{p}}\cdot[(\nabla_{\mathrm{x}}\delta V^{[\alpha^{\mathrm{x}},2]})(\alpha^{\mathrm{x}}+\beta^{\mathrm{x}})]\right|^2 \tag{D17}$$

$$= \int \mathrm{d}\beta\,\tau_{\alpha,\sigma}(\alpha+\beta)\left|(\sigma^{-1}\beta)^{\mathrm{p}}\cdot\frac{1}{2!}[(\beta^{\mathrm{x}}\cdot\nabla_{\mathrm{x}})^2\nabla_{\mathrm{x}}V](\alpha^{\mathrm{x}}+\xi(\beta^{\mathrm{x}})\beta^{\mathrm{x}})\right|^2 \tag{D18}$$

$$= \int \mathrm{d}\beta\,\tau_{\alpha,\sigma}(\alpha+\beta)\left|\frac{1}{2!}[(\beta^{\mathrm{x}}\cdot\nabla_{\mathrm{x}})^2((\beta\sigma^{-1})^{\mathrm{p}}\cdot\nabla_{\mathrm{x}})V](\alpha^{\mathrm{x}}+\xi(\beta^{\mathrm{x}})\beta^{\mathrm{x}})\right|^2 \tag{D19}$$

$$\le \frac{\|\nabla^3 V\|^2}{4}\int \mathrm{d}\beta\,\tau_{\alpha,\sigma}(\alpha+\beta)\left[|\beta^{\mathrm{x}}|^2|(\beta\sigma^{-1})^{\mathrm{p}}|\right]^2 \tag{D20}$$

$$\le \frac{\|\nabla^3 V\|^2}{4}\int \mathrm{d}\beta\,\tau_{\alpha,\sigma}(\alpha+\beta)(\beta^\top P_{\mathrm{x}}\beta)^2(\beta^\top\sigma^{-1}P_{\mathrm{p}}\sigma^{-1}\beta) \tag{D21}$$

$$\le \frac{\|\nabla^3 V\|^2}{4}\left[(\mathrm{Tr}\,\sigma^{\mathrm{xx}})^2\,\mathrm{Tr}(\sigma_{\mathrm{pp}}^{-1}) + 2\,\mathrm{Tr}(\sigma_{\mathrm{pp}}^{-1})\,\mathrm{Tr}[(\sigma^{\mathrm{xx}})^2]\right] \tag{D22}$$

$$= \frac{\|\nabla^3 V\|^2}{4}d^2(d+2)\|\sigma_{\mathrm{pp}}^{-1}\|_{\mathrm{op}}\|\sigma^{\mathrm{xx}}\|_{\mathrm{op}}^2 \tag{D23}$$

where we have again used the Cauchy-Schwartz inequality and the Gaussian integrals reviewed in Appendix E 2. (In particular, $\mathrm{Tr}[\sigma\sigma^{-1}P_{\mathrm{p}}\sigma^{-1}] = \mathrm{Tr}[\sigma_{\mathrm{pp}}^{-1}]$ and $\mathrm{Tr}[\sigma P_{\mathrm{x}}\sigma\sigma^{-1}P_{\mathrm{p}}\sigma^{-1}] = \mathrm{Tr}[P_{\mathrm{x}}P_{\mathrm{p}}] = 0$.) When σ is the covariance of a pure Gaussian quantum state, its the eigenvalues come in pairs λ, $\hbar^2/(4\lambda)$ that are

associated with symplectically conjugate directions (Appendix C). This means $\|\sigma_{\rm pp}^{-1}\|_{\rm op} = (4/\hbar^2)\|\sigma_{\rm xx}\|_{\rm op}$ so that

$$\left\|\delta\mathcal{L}^{(\alpha)}[\tau_{\alpha,\sigma}]\right\|_{L^1}^2 \leq \frac{\|\nabla^3 V\|^2}{\hbar^2} d^2(d+2)\|\sigma^{\rm xx}\|_{\rm op}^3 \tag{D24}$$

which implies (D2) because $d \geq 1$. □

For N particles in n spatial dimensions with k-wise interactions, the dimension is $d = Nn$ but the $d^{\frac{3}{2}}$ dimensional factor in Lemma 1 can be replaced with a factor of only $k^{\frac{3}{2}}n^{\frac{3}{2}}$, a more favorable scaling. One therefore expects extensive error for many-body systems. This growth in error is analogous to that of the orthogonality catastrophe, and therefore a more local notion of error seems to be needed to study the quantum-classical correspondence in many-body systems.

Appendix E: Gaussian derivatives and integrals

1. Gaussian derivatives

The Gaussian probability distribution with mean α and covariance matrix σ is

$$\tau_{\alpha,\sigma}(\alpha+\beta) = \frac{e^{-\beta^\top \sigma^{-1}\beta/2}}{(2\pi)^d\sqrt{\det\sigma}} = \frac{1}{(2\pi)^d\sqrt{\det\sigma}} \exp\left(-\frac{1}{2}\beta^a\sigma_{ab}^{-1}\beta^b\right) \tag{E1}$$

Let us consider this a real-valued function of any vector β and any invertible matrix σ, including non-symmetric ones, so that σ_{ab} and σ_{ba} are independent variables for the purposes of partial derivatives. However, at the end we will evaluate these derivatives on the subspace where σ is symmetric. Recalling our notation $\partial_c = \partial/\partial\beta^c$ so $\partial_c\beta^a = \delta_c{}^a$, we have

$$\partial_d\left(\beta^a\sigma_{ab}^{-1}\beta^b\right) = \sigma_{db}^{-1}\beta^b + \beta^a\sigma_{ad}^{-1} \tag{E2}$$

$$\partial_c\partial_d\left(\beta^a\sigma_{ab}^{-1}\beta^b\right) = \sigma_{dc}^{-1} + \sigma_{cd}^{-1}. \tag{E3}$$

We also deploy the standard [70] matrix derivative identities[71]

$$\frac{\partial \det Z}{\partial y} = (\det Z)\,\mathrm{Tr}\left[Z^{-1}\frac{\partial Z}{\partial y}\right], \tag{E4}$$

$$\frac{\partial Z^{-1}}{\partial y} = -Z^{-1}\frac{\partial Z}{\partial y}Z^{-1} \tag{E5}$$

for an invertible matrix Z, so in particular

$$\frac{\partial \det Z}{\partial Z^{ab}} = (\det Z)Z_{ba}^{-1}, \tag{E6}$$

$$\frac{\partial Z_{cd}^{-1}}{\partial Z^{ab}} = -Z_{ca}^{-1}Z_{bd}^{-1}. \tag{E7}$$

Combining these we get

$$\partial_a\partial_b\tau_{\alpha,\sigma}(\alpha+\beta) = (\sigma_{ac}^{-1}\beta^c\sigma_{bd}^{-1}\beta^d - \sigma_{ab}^{-1})\tau_{\alpha,\sigma}(\alpha+\beta) = 2\frac{\partial}{\partial\sigma^{ab}}\tau_{\alpha,\sigma}(\alpha+\beta), \tag{E8}$$

when evaluated for symmetric σ. (As expected, this is singular when σ is non-invertible.) Weyl quantizing both sides with $\mathrm{Op}_\hbar = \mathcal{W}^{-1}$ gives the corresponding quantum expression $\partial_a\partial_b\hat{\tau}_{\alpha,\sigma} = 2\frac{\partial}{\partial\sigma_{ab}}\hat{\tau}_{\alpha,\sigma}$.

2. Gaussian integrals

Here we recall the evaluation of some Gaussian integrals, as can be done with Wick's theorem. We define the shorthand:

$$\begin{aligned}\langle(\beta^\top A\beta)\rangle_\sigma :=& \int \mathrm{d}\beta \tau_{0,\sigma}(\beta)(\beta^\top A\beta)\\ =& \int \mathrm{d}\beta \tau_{\alpha,\sigma}(\alpha+\beta)(\beta^\top A\beta)\\ =& A_{ab}\int \mathrm{d}\beta \tau_{\alpha,\sigma}(\alpha+\beta)\beta^a\beta^b\\ =& A_{ab}\sigma^{ab}\\ =& \operatorname{Tr}[\sigma A].\end{aligned} \tag{E9}$$

for any positive semidefinite matrix A. (The covariance matrix σ is also positive semidefinite, of course.) Likewise, for B and C also positive semidefinite, we have

$$\begin{aligned}\langle(\beta^\top A\beta)(\beta^\top B\beta)\rangle_\sigma :=& \int \mathrm{d}\beta \tau_{\alpha,\sigma}(\alpha+\beta)(\beta^\top A\beta)(\beta^\top B\beta)\\ =& A_{ab}B_{cd}\left[\sigma^{ab}\sigma^{cd}+2\sigma^{ad}\sigma^{bc}\right]\\ =& \operatorname{Tr}[\sigma A]\operatorname{Tr}[\sigma B]+2\operatorname{Tr}[\sigma A\sigma B]\end{aligned} \tag{E10}$$

and

$$\begin{aligned}\langle(\beta^\top A\beta)(\beta^\top B\beta)(\beta^\top C\beta)\rangle_\sigma :=& \int \mathrm{d}\beta \tau_{\alpha,\sigma}(\alpha+\beta)(\beta^\top A\beta)(\beta^\top B\beta)(\beta^\top C\beta)\\ =& A_{ab}B_{cd}C_{ef}\Big[\sigma^{ab}\sigma^{cd}\sigma^{ef}+2\left(\sigma^{ab}\sigma^{cf}\sigma^{de}+\sigma^{af}\sigma^{cd}\sigma^{be}+\sigma^{ad}\sigma^{bc}\sigma^{ef}\right)\\ &+4\left(\sigma^{ad}\sigma^{be}\sigma^{cf}+\sigma^{af}\sigma^{bc}\sigma^{de}\right)\Big]\\ =& \operatorname{Tr}[\sigma A]\operatorname{Tr}[\sigma B]\operatorname{Tr}[\sigma C]+2\operatorname{Tr}[\sigma A]\operatorname{Tr}[\sigma B\sigma C]+2\operatorname{Tr}[\sigma B]\operatorname{Tr}[\sigma C\sigma A]\\ &+2\operatorname{Tr}[\sigma C]\operatorname{Tr}[\sigma B\sigma A]+8\operatorname{Tr}[\sigma A\sigma B\sigma C].\end{aligned} \tag{E11}$$

Appendix F: Applications to physical examples

Previous studies have estimated the strength of the environment-induced diffusion for real physical systems: a grain of dust being decohered by the cosmic microwave background, a large molecule being decohered by sunlight, and so on.

We encounter a difficulty when plugging these physical estimates into our bound. In many naturally occurring interactions between system and bath, only the position variables are coupled to the bath, with no direct coupling to momentum. Thus $D_{(\mathrm{x})} = \hbar|\ell|_{\mathrm{p}}^2 = 0$ while $D_{(\mathrm{p})} = \hbar|\ell_{\mathrm{x}}|^2 \neq 0$, i.e., there is diffusion in momentum but not position. On the other hand, if we unpack the constants in Eq. (8),

$$\epsilon = t\hbar^2 d^{\frac{3}{2}} \frac{\|\nabla^3 V\|}{(\|\nabla^2 V\| m)^{\frac{3}{4}}} \max\left\{\frac{1/m}{D_{(\mathrm{x})}}, \frac{\|\nabla^2 V\|}{D_{(\mathrm{p})}}, \frac{1}{\hbar}\right\}^{\frac{3}{2}}, \tag{F1}$$

one can see our bound deteriorates if either $D_{(\mathrm{x})}$ or $D_{(\mathrm{p})}$ become too small. At first glance, this problem may seem irreparable. The diffusion in position variables, which is missing in most natural models of system-bath interactions, corresponds to decohering superpositions in the system's momentum variables. Yet if superpositions of momentum are allowed to persist, why should we expect classical behavior?

Physically, the remedy is that even if a superposition $|\psi_1\rangle + |\psi_2\rangle$ of two coherent states with different momenta but overlapping positions is not *immediately* decohered by purely positional decoherence, they will ultimately evolve into a superposition of two states with different positions as well, at which point the positional decoherence takes effect. Thus heuristically, we can model an effective positional diffusion $D_{(\mathrm{x}),\mathrm{eff}}$

as $D_{(\mathrm{x}),\mathrm{eff}} \propto D_{(\mathrm{p})}$ even when $D_{(\mathrm{x})}$ is actually zero. In particular, simple dimensional analysis suggests we can model $D_{(\mathrm{x}),\mathrm{eff}}$ as

$$D_{(\mathrm{x}),\mathrm{eff}} \sim \frac{D_{(\mathrm{p})}}{\|\nabla^2 V\| m}. \tag{F2}$$

In that case, we ignore the first argument in the $\min\{\cdot\}$ of (F1), or equivalently take $D_0 = D_{(\mathrm{p})}\tau_H/p_H^2$ in (6). We emphasize this substitution is heuristic, though it is justified by work in progress.

With this substitution, we can apply our bound to examples using Table 3.1 of [72], which provides estimates for the "localization rate" $\Lambda_{(\mathrm{x})} \sim \hbar^{-2} D_{(\mathrm{p})}$, quantifying positional decoherence. As an example, we consider a dust particle with diameter 10μm and mass 10^{-11} kg decohering in the sunlight, for which $\Lambda_{(\mathrm{x})} \sim 10^{25}\mathrm{m}^{-2}\mathrm{s}^{-1}$.

Consider a particle with characteristic energy $E \sim mv^2$ for characteristic velocity v, and characteristic length scale s for the variation of the potential. That is, we take $\|\nabla^2 V\| \sim Es^{-2}$ and $\|\nabla^3 V\| \sim Es^{-3}$. Then we have

$$\|\widetilde{\rho}(t) - \rho(t)\|_{\mathrm{Tr}} \lesssim t\hbar^{-1} v^{\frac{7}{2}} m \Lambda_{(\mathrm{x})}^{-\frac{3}{2}} s^{-\frac{9}{2}}, \tag{F3}$$

i.e. the bound is useful for time

$$t \lesssim \hbar v^{-\frac{7}{2}} m^{-1} \Lambda^{\frac{3}{2}} s^{\frac{9}{2}}. \tag{F4}$$

For the 10 micron dust particle in sunlight, with characteristic velocity $v = 1\mathrm{m/s}$ in a potential varying on the length scale of $s = 1\,\mathrm{m}$, the error $\|\rho(t) - \tilde{\rho}(t)\| \ll 1$ is small for three million years.